PROBLEMS IN COMPLEX ANALYSIS

These problems have been collected them from a number of text books.
I thank Sourav Ghosh for pointing out several errors in previous version.

Notation: $U = \{z : |z| < 1\}$ and $T = S^1 = \{z : |z| = 1\}$. Differentiability is in the complex analytic sense (not as function of two real variables). Definition: f is analytic or holomorphic on an open set if it is differentiable at each point. $H(\Omega)$ is the class of all holomorphic functions on Ω. $\overset{uc}{\to}$ stands for uniform convergence on compact sets. f^n is the $n-th$ power of f. The $n-th$ derivative is denoted by $f^{(n)}$ and $n-th$ iterate by $f_{(n)}$.

Principal logarithm is defined by $Log(z) = \log|z| + i\theta$ where $-\pi < \theta \leq \pi$ for $z \in \mathbb{C}\backslash\{0\}$. $B(z,r)$ denotes $\{\zeta : |\zeta - z| < r\}$.

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If you have comments or find errors please send a mail to this address.

1. Find a sequence of complex numbers $\{z_n\}$ such that $\sin z_n$ is real for all n and $\to \infty$ as $n \to \infty$?

2. At what points is $f(z) = |z|$ differentiable? At what points is $f(z) = |z|^2$ differentiable?

3. If f is a differentiable function from a region Ω in $\mathbb{C}$ into $\mathbb{R}$ prove that f is necessarily a constant.

4. Find all entire functions f such that $f^n(z) = z$ for all z, n being a given positive integer.

5. If f and $\bar{f}$ are both analytic in a region Ω show that they are constants on Ω.

6. If f^2 and $(\bar{f})^5$ are analytic in a region show that f is a constant on that region.

7. If f is analytic in a region Ω and if $|f|$ is a constant on Ω show that f is a constant on Ω.

8. Prove that Log is not continuous on $\mathbb{C}\backslash\{0\}$.

9. Prove that the function *Log* is differentiable on $\Omega = \mathbb{C}\backslash\{x \in \mathbb{R} : x \leq 0\}$. Find its derivative and prove that there is no power series $\sum\limits_{n=0}^{\infty} a_n(z-c)^n$ convergent in $\mathbb{C}\backslash\{x \in \mathbb{R} : x \leq 0\}$ whose sum is *Log*.

10. Let p be a non-constant polynomial, $c > 0$ and $\Omega = \{z : |p(z)| < c\}$. Prove that $\partial\Omega = \{z : |p(z)| = c\}$ and that each connected component of Ω contains a zero of p.

11. Prove that there is no differentiable function f on $\mathbb{C}\backslash\{0\}$ such that $e^{f(z)} = z$ for all $z \in \mathbb{C}\backslash\{0\}$.

Hint: If it exists, compare it with *Log*.

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12. Let γ be a piecewise continuously differentiable map : $[0,1] \to \mathbb{C}$ and $h : \gamma^* \to \mathbb{C}$ be continuous (γ^* is the range of γ). Show that $f(z) = \int\limits_{\gamma} \frac{h(\zeta)}{\zeta - z} d\zeta$ defines a holomorphic function on $\mathbb{C}\backslash\gamma^*$.

13. If γ is a piecewise continuously differentiable map : $[0,1] \to \mathbb{C}$ show that the total variation of γ is $\int\limits_0^1 |\gamma'(t)|\, dt$.

14. If p is a polynomial and if the maximum of $|p|$ on a region Ω is attained at an interior point a show, without using The Maximum Modulus Principle, that p is a constant.

Hint: Compute the integral of $\frac{p(z)}{z-a}$ over a circle with centre a contained in Ω.

15. If $f(x+iy) = \sqrt{|xy|}$ show that f is not differentiable at 0 even though Cauchy-Riemann equations are satisfied at 0.

16. Show that $\log\sqrt{x^2+y^2}$ is a harmonic function on $\mathbb{C}\backslash\{0\}$ which is not the real part of any holomorphic function.

17. If f is holomorphic on a region Ω and e^f is constant on Ω show that f is constant on Ω.

18. If f is an entire function and $\operatorname{Re} f$ (or $\operatorname{Im} f$) is bounded above or below show that f is constant.

19. Prove that $\frac{|a-b|}{\left|1-\bar{a}b\right|} \geq \frac{|a|-|b|}{1-|ab|}$ if either $|a|$ and $|b|$ are both less than 1 or both greater than 1.

20. If $f : U \to U$ is holomorphic show that $\frac{|f(\beta)-f(\alpha)|}{\left|1-\bar{f}(\beta)f(\alpha)\right|} \leq \frac{|\beta-\alpha|}{\left|1-\bar{\beta}\alpha\right|}$ for all $\alpha, \beta \in U$.

21. Prove that a holomorphic function from U into itself has atmost one fixed point unless it is the identity map.

22. If f is a conformal equivalence of U show that f maps open balls in U onto open balls.

23. Let Ω be a region, $f \in C(\Omega)$ and let f^n be holomorphic in Ω for some positive integer n. Show that f is holomorphic in Ω.

24.

a) If f is an entire function such that $|f(z)| \leq 1 + \sqrt{|z|}$ for all $z \in \mathbb{C}$ show that f is a constant.

b) If f is an entire function such that $|f(z)| \leq M\,|z|^N$ for $|z|$ sufficiently large (where M is a positive constant) show that f is a polynomial.

25. Find the largest open set on which $\int\limits_0^1 \frac{1}{1+tz}dt$ is analytic. Do the same for $\int\limits_0^\infty \frac{e^{tz}}{1+t^2}dt$.

26. If f and g are holomorphic functions on a region Ω with no zeros such that $\{z : \frac{f'}{f}(z) = \frac{g'}{g}(z)\}$ has a limit point in Ω find a simple relation between f and g.

27. If f is a holomorphic function on a region Ω which is not identically zero show that the zeros of the function form an atmost countable set.

28. Is Mean Value Theorem valid in the complex case? (i.e., if f is analytic in a convex region and z_1, z_2 are two points in the region can we always find a point ζ on the line segment from z_1 to z_2 such that $f(z_2) - f(z_1) = f'(\zeta)(z_2 - z_1)$?)

29. Let f be holomorphic on a region Ω with no zeros. If there is a holomorphic function h such that $h' = \frac{f'}{f}$ show that f has a holomorphic logarithm on Ω (i.e. there is a holomorphic function H such that $e^H = f$. Show that h need not exist and give a sufficient condition on Ω that ensures existence of h.

30. Prove that a bounded harmonic function on $\mathbb{R}^2$ is constant.

31. If f is an entire function such that $|f(z)| \geq M\,|z|^n$ for $|z| \geq R$ for some $n \in \mathbb{N}$ and some M and R in $(0, \infty)$ show that f is a polynomial whose degree is at least n.

32. If f is an entire function which is not a constant prove that $\max\{|f(z)| : |z| = r\}$ is an increasing function of r which $\to \infty$ as $r \to \infty$.

33. If $f \in C(\bar{U}) \cap H(U)$ and $f(z) = 0$ on $\{e^{i\theta} : a < \theta < b\}$ for some $a < b$ show that f is identically 0.

34. True or false: if f and g are entire functions such that $f(z)g(z) = 1$ for all z then f and g are constants. [What is the answer if f and g are polynomials?]

35. If $f : U \to U$ is holomorphic, $a \in U$ and $f(a) = a$ prove that $|f'(a)| \leq 1$.

36. The result of Problem 35 holds if U is replaced by any region that is conformally equivalent to U. [A conformal equivalence is a bijective biholomorphic map].

37. According to Riemann Mapping Theorem, any simply connected region other than $\mathbb{C}$ is conformally equivalent to U. Hence, previous problem applies to any such region. Is the result valid for $\mathbb{C}$?

38. Prove that only entire functions that are one-to-one are of the type $f(z) = az + b$.

39. Prove that $\{z : 0 < |z| < 1\}$ and $\{z : r < |z| < R\}$ are not conformally equivalent if $r > 0$.

40. Let $0 < r_1 < R_1$ and $0 < r_2 < R_2$. Prove that $\{z : r_1 < |z| < R_1\}$ and $\{z : r_2 < |z| < R_2\}$ are conformally equivalent $\Leftrightarrow \frac{R_1}{r_1} = \frac{R_2}{r_2}$

41. Show that if a holomorphic map f maps U into itself it need not have a fixed point in U. Even if it extends to a continuous map of the closure of U onto itself the same conclusion holds.

42. If f is holomorphic on U, continuous on the closure of U and $|f(z)| < 1$ on T prove that f has at least one fixed point in U. Can it have more than one fixed point?

43. If f is holomorphic : $U \to U$ and $f(0) = 0$ and if $\{f_n\}$ is the sequence of iterates of f ($i.e. f_1 = f, f_{n+1} = f \circ f_n, n \geq 1$) prove that the sequence $\{f_n\}$ converges uniformly on compact subsets of U to 0 unless f is a rotation.

44. Let f be a homeomorphism of $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ (with the metric induced by the stereographic projection). Assume that f is differentiable at all points of $\mathbb{C} \cup \{\infty\}$ except $f^{-1}\{\infty\}$. Prove that f is a Mobius Transformation.

45. Prove that the only conformal equivalences : $U \backslash \{0\} \overset{onto}{\rightarrow} U \backslash \{0\}$ are rotations.

46. Prove that $\pi \cot \pi z = \frac{1}{z} + \sum\limits_{n=1}^{\infty} \frac{2z}{z^2 - n^2}$ if z is not an integer.

47. Prove or disprove: $Log(z_1 z_2) = Log(z_1) + Log(z_2)$

48.
a) Discuss convergence of the following infinite products:
$\prod\limits_{n=1}^{\infty} \frac{1}{n^p} (p > 0), \prod\limits_{n=1}^{\infty} (1 + \frac{i}{n}), \prod\limits_{n=1}^{\infty} \left|1 + \frac{i}{n}\right|.$

49. Let $a_n \neq 0$ for all n. Prove that $\prod\limits_{n=1}^{\infty} [1 + |1 - a_n|]$ converges if and only if $\sum\limits_{n=1}^{\infty} |Log(a_n)| < \infty$.

50. Prove or disprove the following:
$\sum\limits_{n=1}^{\infty} |Log(a_n)| < \infty \Leftrightarrow \sum\limits_{n=1}^{\infty} |1 - a_n| < \infty$ and $\sum\limits_{n=1}^{\infty} Log(a_n)$ is convergent $\Leftrightarrow$ $\sum\limits_{n=1}^{\infty} [1 - a_n]$ is convergent.

51. Prove that $\prod\limits_{n=1}^{\infty} z_n$ converges $\Leftrightarrow \sum Log(z_n)$ converges. Use this to prove that $\prod\limits_{n=1}^{\infty} (1 + i/n)$ is not convergent.

52. Prove that $\sin \pi z = \pi z \prod\limits_{n=1}^{\infty} (1 - \frac{z^2}{n^2})$

53. Let $B(z) = \prod\limits_{n=1}^{\infty} \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \bar{a}_n z}$. Prove that if $0 < |a_n| < 1$ and $\sum [1 - |a_n|] < \infty$ then the product converges uniformly on compact subsets of U and that $B(z)$ is a holomorphic function on this disk with zeros precisely at the points $a_n, n = 1, 2, ...$ Prove that $\{a_n\}$ can be chosen so that every point of T is a

limit point of $\{a_n\}$; prove that T is a natural boundary of B in this case (in the sense B cannot be extended to a holomorphic function on any larger open set.

54. Say that a function $f : \mathbb{R} \to \mathbb{R}$ is analytic if for each $a \in \mathbb{R}$ there exists $\delta_a > 0$ such that f has a power series expansion on $(a - \delta_a, a + \delta_a)$. Show that the zeros of an analytic function on $\mathbb{R}$ have no limit points unless the function is identically 0.

55. If $f : \mathbb{C} \to \mathbb{C}$ has power series expansion around each point then it has a single power series expansion valid on all of $\mathbb{C}$. Is it true that if $f : \mathbb{R} \to \mathbb{R}$ has power series expansion around each point then it has a single power series expansion valid on all of $\mathbb{R}$?

56. Does there exist an entire function f such that $|f(z)| = |z|^2 e^{\mathrm{Im}(z)}$ for all z? If so, find all such functions. Do the same for $|f(z)| = |z| e^{\mathrm{Im}(z)\,\mathrm{Re}(z)}$.

57. Does there exist a holomorphic function f on U such that $\{f(\frac{1}{n})\} = \{\frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, ...\}$, i.e. $f(\frac{1}{n}) = \frac{1}{n}$ if n is even and $f(\frac{1}{n}) = \frac{1}{n+1}$ if n is odd?

58. If the radius of convergence of $\sum\limits_{n=0}^{\infty} a_{n,k}(z-a)^n$ exceeds R for each k and $\sum\limits_{n=0}^{\infty} a_{n,k}(z-a)^n \to 0$ uniformly on $\{z : |z - z_0| = r\}$ show that it converges uniformly on $\{z : |z - z_0| \leq r\}$ provided $R > r + |z_0 - a|$.

59. Let f be continuous and bounded on $\{z : |z| \leq 1\} \backslash F$ where F is a finite subset of T. If f is holomorphic on U and $|f(z)| \leq M$ on $\partial U \backslash F$ show that $|f(z)| \leq M$ on U.

Remark: a similar result is valid in a general bounded region. See problem 263.

60. Let $\Omega = \{z : \mathrm{Re}(z) > 0\}$. If f is continuous on the closure of Ω, holomorphic on Ω and if $|f(z)| \leq 1$ on $\partial\Omega$ does it follow that the same inequality holds on Ω?.

61. Let $\Omega = \{z : a < \mathrm{Im}(z) < b\}, f \in H(\Omega)$ and f be continuous on the closure of Ω. Prove that if $|f(z)| \leq 1$ on $\partial\Omega$ then the same inequality holds on Ω.

62. Prove that $f(z) = \frac{z}{(1-z)^2}$ is one-to-one on U and find the image of U.

64. Evaluate $\int\limits_\gamma \frac{1}{(z-2)(2z+1)^2(3z-1)^3}dz$ and $\int\limits_\gamma \frac{1}{(z-10)(z-\frac{1}{2})^{100}}dz$ where $\gamma(t) = 12e^{2\pi it}(0 \leq t \leq 1)$

65. Find the number of zeros of $z^7 + 4z^4 + z^3 + 1$ in U and the annulus $\{1 < |z| < 2\}$.

67. Let $1 < a < \infty$. prove that $z + a - e^z$ has exactly one zero in the left half plane $\{z : \mathrm{Re}(z) < 0\}$.

68. If $0 < |a| < 1$ show that the equation $(z-1)^n e^z = a$ has exactly n solutions in $\mathrm{Re}\, z > 0$. Prove that all the roots are simple roots. If $|a| \leq \frac{1}{2^n}$ prove that all the roots are in $\{z : |z-1| < \frac{1}{2}\}$.

69. Prove that $f(z) = 1 + z^2 + z^{2^2} + ... + z^{2^n} + ...$ has U as its *natural boundary* in the sense it cannot be extended to a holomorphic function on any region which properly contains U.

70. If p is a polynomial such that $|p(z)| = p(|z|)$ for all z prove that $p(z) = cz^n$ for some $c \geq 0$ and some $n \in \mathbb{N} \cup \{0\}$.

71. Prove that above result holds if p is replaced by an entire function f.

72. Prove the two dimensional Mean value Property:
the average of a holomorphic function over an open ball is the value at the centre.

73. Construct a conformal equivalence between the first quadrant and the upper half plane. Also, find a conformal equivalence between U and its intersection with the right half plane.

74. Find a conformal equivalence between the sector $\{z \neq 0 : \theta_1 < \arg(z) < \theta_2\}$ with $0 < \theta_1 < \theta_2 < \pi/2$ and U.

75. Prove that if γ is a closed path in a region Ω and $f \in H(\Omega)$ then $\mathrm{Re}(\int_\gamma f\bar(z) f'(z)dz) = 0$.

76. Prove or disprove: given any sequence $\{a_n\}$ of complex numbers there is a holomorphic function f in some neighbourhood of 0 such that $f^{(n)}(0) = a_n$ for all n.

77. If f is holomorphic on $\Omega\backslash\{a\}$ prove that $e^{f(z)}$ cannot have a pole at a.

78. Prove that $\int\limits_0^{2\pi} \log\left|1-e^{i\theta}\right| d\theta = 0$.

79. Use above result to prove Jensen's Formula:
If $f \in H(B(0,R)), f(0) \neq 0, 0<r<R$ and $\alpha_1, \alpha_2, ..., \alpha_N$ are the zeros of f in $\bar{B}(0,r)$ listed according to multiplicities then $|f(0)| \prod\limits_{n=1}^{N} \frac{r}{|\alpha_n|} = e^{\frac{1}{2\pi}\int\limits_0^{2\pi} \log\left|f(re^{i\theta})\right| d\theta}$.

Also prove Jensen's inequality: $\log|f(0)| \leq \frac{1}{2\pi}\int\limits_0^{2\pi} \log\left|f(re^{i\theta})\right| d\theta$.

80. Let Ω be an open set containing 0 and $f \in H(\Omega)$. Prove that $\bar{f}(z) = f(\bar{z})$ for all z with $|z|$ sufficiently small $\Leftrightarrow f^{(n)}(0) \in \mathbb{R}$ for all $n \geq 0$.

81. If $f \in H(U), f(0) = 0, f'(0) \neq 0$ prove that there is no $g \in H(U\backslash\{0\})$ such that $g^2 = f$.

82. If f is an entire function such that $|f(z)| \to \infty$ as $|z| \to \infty$ prove that $|f(z)| \geq c|z|$ for some positive number c for all z with $|z|$ sufficiently large. Hence prove that f is a polynomial.

83. Let Ω be a region, $\{f_n\} \subset H(\Omega)$ and assume that $\{f_n\}$ is uniformly bounded on each compact subset of Ω. Let C be the set of points where $\{f_n\}$ is convergent. If this set has a limit point in Ω prove that $\{f_n\}$ converges uniformly on compact subsets of Ω to a holomorphic function.

84. Let Ω be a region in $\mathbb{C}$. Let $f : \Omega \to \mathbb{C}$ be continuous and have first continuous partial derivatives at all points. Suppose $\lim\limits_{h\to 0}\left|\frac{f(z+h)-f(z)}{h}\right|$ exists and $\in (0,\infty)$ for each $z \in \Omega$. Show that either f is holomorphic in Ω or $\bar{f}$ is holomorphic in Ω.

85. Give an example of a function f which is continuous on a closed strip, holomorphic in the interior, bounded on the boundary but not bounded on the strip! [See also problem #61 above].

86. Let $u(z) = \operatorname{Im}\{(\frac{1+z}{1-z})^2\}$. Show that u is harmonic in U and $\lim\limits_{r\to 1} u(re^{i\theta}) = 0$ for all θ. Why doesn't this contradict the Maximum Modulus Principle for harmonic functions?

87. If $\phi(|z|)$ is harmonic in the region $\{z : \mathrm{Re}(z) > 0\}$ where ϕ is real valued and "smooth" prove that $\phi(t) \equiv a \log t + b$ for some a and b.

88. Let $f : \bar{U} \to \mathbb{C}$ be a continuous function which is harmonic in U. Prove that f is holomorphic in U if and only if $\int\limits_{-\pi}^{\pi} f(e^{it})e^{int}dt = 0$ for all positive integers n.

89. Let $\Omega = \{z : \mathrm{Im}(z) > 0\}$. If $f : \partial\Omega \equiv \mathbb{R} \to \mathbb{C}$ is bounded and continuous on $\partial\Omega$ show that it is the restriction of a continuous function on $\bar{\Omega}$ which is harmonic in Ω.

90. Prove that the square of a real harmonic function (in a simply connected region) is not harmonic unless it is a constant. When is the product of two real harmonic functions harmonic? Find all holomorphic functions f such that $|f|^2$ is harmonic.

91.

If Ω is a simply connected region, $f : \Omega \to \mathbb{C}$ is harmonic and if f^2 is also harmonic show that either f is holomorphic or $\bar{f}$ is holomorphic in Ω.

92. If u is a real non- constant harmonic in a region Ω prove that the zeros of the gradient of u in Ω have no limit point.

93. If u is harmonic in a region Ω prove that partial derivatives of u of all orders are harmonic.

94. Let $S = \{x \in \mathbb{R} : a \leq x \leq b\}$. Let Ω be a region containing S. Prove that if $f \in H(\Omega \backslash S) \cap C(\Omega)$ then $f \in H(\Omega)$.

95. Let $f, f_n (n = 1, 2, ...)$ be holomorphic functions on a region Ω. If $\mathrm{Re}(f_n) \overset{uc}{\to} \mathrm{Re}(f)$ show that $f_n \overset{uc}{\to} f$. [$\overset{uc}{\to}$ stands for uniform convergence on compact subsets].

96. Let $f(z) = \int\limits_{-1}^{1} \frac{1}{t-z} dt, z \in \mathbb{C} \backslash [-1, 1]$. Prove that f is holomorphic, its imaginary part is bounded, but the real part is not. Prove that $\lim\limits_{z \to \infty} z f(z)$ exists and find this limit. Find a bounded non-constant holomorphic function on $\mathbb{C} \backslash [-1, 1]$.

97. Give an example of a region Ω such that Ω^c is infinite and every bounded holomorphic function on Ω is a constant.

98. If Ω is any region contained in $\mathbb{C}\backslash(-\infty, 0]$ show that there exists a bounded non-constant holomorphic function on Ω.

More generally if there is a non-constant holomorphic function ϕ on Ω such that $\phi(\Omega)$ is contained in $\mathbb{C}\backslash(-\infty, 0]$ the same conclusion holds.

99. If Ω is $\mathbb{C}\backslash(-\infty, 0]$ or a horizontal strip or a vertical strip or $\bar{U}^c$ show that there exist non-constant bounded holomorphic functions on Ω.

100. Prove that there is no holomorphic function f on $\bar{U}^c$ such that $|f(z)| \to \infty$ as $|z| \to 1$.

101. Prove that there is no homeomorphism ϕ from $\bar{\Omega}$, where $\Omega = \{z : \operatorname{Re}(z) > 0\}$, onto $\bar{U}$ which maps Ω onto U and is holomorphic in Ω.

102. Let Ω be a bounded region, $f \in C(\bar{\Omega}\ ) \cap H(\Omega)$ and assume that $|f|$ is a non-zero constant on $\partial\Omega$. If f is not a constant on Ω show that f has at least one zero in Ω.

103. Let f be a non-constant entire function. Prove that the closure of $\{z : |f(z)| < c\}$ coincides with $\{z : |f(z)| \leq c\}$ for all $c > 0$.

104. Prove that if $f \in H(\Omega), [a, b] \subset \Omega$ (where $[a, b]$ is the line segment from a to b) then $|f(b) - f(a)| \leq |b - a| \, |f'(\xi)|$ for some $\xi \in [a, b]$. Also prove that $|f(b) - f(a) - (b - a)f'(a)| \leq \frac{|b-a|^2}{2} \, |f''(\eta)|$ for some $\eta \in [a, b]$.

105. Evaluate $\int\limits_{\gamma} \frac{z^2+1}{z(z^2+4)} dz$ where $\gamma(t) = re^{2\pi it} (0 \leq t \leq 1)$ where $0 < r < 2$. Also compute the same integral for $r > 2$.

106. Give an example of a bounded holomorphic function f on $\mathbb{C}\backslash\mathbb{R}$ which cannot be extended to any larger open set.

107. If $f \in H(0 < |z| < R)$ and $\int\limits_{0<x^2+y^2<R} |f(x + iy)| \, dxdy < \infty$ prove that f has either a removable singularity or a pole of order one at 0.

108. In the previous problem if $\int\limits_{0<x^2+y^2<R} |f(x + iy))|^2 \, dxdy < \infty$ prove that f has a removable singularity at 0.

109. Show that there is no function $f \in H(U) \cap C(\bar{U})$ such that $f(z) = \frac{1}{z} \forall z \in \partial U$.

110. If $f \in C(U), f_n \in H(U)$ and $f_n \to f$ in $L^1(U)$ then $f \in H(U)$.

111. Any conformal equivalence of $\mathbb{C}\backslash\{0\}$ is of the form cz or of the form $\frac{c}{z}$ where c is a constant.

112. If $x_1 > x_2 > x_3 > ..., \{x_n\} \to 0$ and $f \in H(U)$ with $f(x_n) \in \mathbb{R} \forall n$ then $f^{(k)}(0) \in \mathbb{R} \forall k$.

113. Let $\{f_n\} \subset H(D)$ where D is an open disc. Assume that $f_n(D) \subset D \backslash \{0\} \forall n$ and that $\lim\limits_{n\to\infty} f_n(a) = 0$ where a is the center of D. Then $\lim\limits_{n\to\infty} f_n(z) = 0$ uniformly on compact subsets of D.

114. Let $\{u_n\}$ be a sequence of (strictly) positive harmonic functions on a simply connected region Ω such that $\sum u_n = \infty$ at one point. Then the series diverges at every point of some disc around that point. On the other hand, if it converges at one point it converges uniformly on compact subsets of Ω.

115. Find all limit points of the sequence $\{\frac{1}{n}\sum\limits_{k=1}^{n} k^{ia}\}_{n=1,2,...}$ where a is a non-zero real number.

116. Let f have an isolated singularity at a point a. Prove that e^f cannot have a pole at a.

117. Let f be holomorphic on U and assume that for each $r \in (0,1)$, $f(re^{it})$ has a constant argument (i.e. $f(re^{it}) = \left|f(re^{it})\right| e^{ia_r}$ where the real number a_r does not depend on t. Show that f is a constant.

118. [based on problem 117)] Let $f \in H(\Omega)$ and suppose $|f|$ is harmonic in Ω. Show that f is a constant.

119. Let $f \in H(U), f(U) \subset U, f(0) = 0$ and $f(\frac{1}{2}) = 0$. Show that $|f'(0)| \leq \frac{1}{2}$. Give an example to show that equality may hold.

120. Let $f \in H(U), f(U) \subset U, f(0) = 0, f'(0) = 0, f''(0) = 0..., f^{(k)}(0) = 0$ where k is a positive integer. Show that $\left|f(\frac{1}{2})\right| \leq \frac{1}{2^k}$ and find a necessary and sufficient condition that $\left|f(\frac{1}{2})\right| = \frac{1}{2^k}$.

121. If f and $zf(z)$ are both harmonic in a region Ω then f is analytic in Ω.

122. Prove that $f(re^{i\theta}) = \sum\limits_{n=-\infty}^{\infty} r^{|n|} \sin(n\alpha) e^{in\theta}$ is harmonic in U.

123. If $\Omega = \{z : \text{Re}(z) > 0\}$ and f is a bounded holomorphic function on Ω with $f(n) = 0 \forall n \in \mathbb{N}$ show that $f(z) = 0 \ \forall z \in \Omega$.

124. Show that there is a holomorphic function f on $\{z : \mathrm{Re}(z) > -1\}$ such that $f(z) = \frac{z^2}{2} - \frac{z^3}{(2)(3)} + \frac{z^4}{(3)(4)} - \ldots$ for $|z| < 1$.

125. Consider the series $z + \frac{z^2}{2} + \frac{z^3}{3} + \ldots$ on U and $i\pi - (z-2) + \frac{(z-2)^2}{2} - \frac{(z-2)^3}{3} + \ldots$ on $\{z : |z-2| < 1\}$. (These two regions are disjoint). Show that there is a region Ω and a function $f \in H(\Omega)$ such that Ω contains both U and $\{z : |z-2| < 1\}$, $f(z) = z + \frac{z^2}{2} + \frac{z^3}{3} + \ldots$ on U and $f(z) = i\pi - (z-2) + \frac{(z-2)^2}{2} - \frac{(z-2)^3}{3} + \ldots$ on $\{z : |z-2| < 1\}$.

126. Let $f : U \to U$ be holomorphic with $f(0) = 0 = f(a)$ where $a \in U \backslash \{0\}$. Show that $|f'(0)| \leq |a|$.

127. Prove that a complex valued function u on a simply connected region Ω is harmonic if and only if it is of the form $f + \bar{g}$ for some $f, g \in H(\Omega)$.

128. Let $f(z) = z + \frac{1}{z}(z \in \mathbb{C}\backslash\{0\})$. Show that $f(\{z : 0 < |z| < 1\}) = f(\{z : |z| > 1\}) = \mathbb{C}\backslash[-2,2]$ and that $f(\{z : |z| = 1\}) = [-2,2]$. Show also that f is conformal equivalence of both the regions $\{z : 0 < |z| < 1\}$)and $\{z : |z| > 1\}$ with $\mathbb{C}\backslash[-2,2]$. Prove that $\{z : |z| > 1\}$ is not simply connected.

129. Show that there is no bounded holomorphic function f on the right-half plane which is 0 at the points $1, 2, 3, \ldots$ and 1 at the point $\sqrt{2}$. What is the answer if 'bounded' is omitted?

130. Prove or disprove: if $\{a_n\}$ has no limit points and $\{c_n\} \subset \mathbb{C}$ then there is an entire function f with $f(a_n) = c_n \forall n$.

131. Let Ω be a bounded region, $f \in H(\Omega)$ and $\limsup_{z \to a} |f(z)| \leq M$ for every point a on the boundary of Ω.Show that $|f(z)| \leq M$ for every $z \in \Omega$.

132. Let f be an entire function such that $\frac{f(z)}{z} \to 0$ as $|z| \to \infty$. Show that f is a constant.

133. Let f be an entire function which maps the real axis into itself and the imaginary axis into itself. Show that $f(-z) = -f(z) \forall z \in \mathbb{C}$. Is the converse true?

134. Let f be a continuous function : $\mathbb{C} \to \mathbb{C}$ such that $f(z^2 + 2z - 6)$ is an entire function. Show that f is an entire function.

135. If f and g are entire functions with no common zeros and if h is an entire function show that $h = fF + gG$ for some entire functions F and G.

136. Show that the series $\sum_{n=1}^{\infty} \frac{z^n}{n}$ converges if $|z| \leq 1$ and $z \neq 1$.

137. Show that the convergence of $\sum_{n=1}^{\infty} \frac{\sin(nz)}{n}$ implies that $z \in \mathbb{R}$.

138. If $f \in C(\overline{U}) \cap H(U)$ and f is real valued on $T = \partial U$ then f is a constant.

139. Let $\Omega = \{z : \text{Im}(z) > 0\}$ and $f \in H(\Omega) \cap C(\overline{\Omega})$. If $f(x) = x^4 - 2x^2$ for $0 < x < 1$ find $f(i)$.

140. Let Ω be a region and m denote Lebesgue measure on Ω. If $\{f_n\} \subset H(\Omega) \cap L^2(\Omega)$ and if $\{f_n\}$ converges in $L^2(\Omega)$ to f show that $f \in H(\Omega)$.

141. Let Ω be a region containing $\overline{U}$ and $f \in H(\Omega)$. If $|f(z)| = 1$ whenever $|z| = 1$ show that $U \subset f(\Omega)$.

142. Let Ω be a bounded region, $f, g : \overline{\Omega} \to \mathbb{C}$ be continuous and holomorphic in Ω. If $|f(z) - g(z)| < |f(z)| + |g(z)|$ on $\partial\Omega$ show that f and g have the same number of zeros in Ω.

143. Let Ω be a bounded region $f : \overline{\Omega} \to \overline{U}$ be continuous and $f \in H(\Omega)$, f not a constant. If $|f(z)| = 1$ whenever $z \in \partial\Omega$ show that $U = f(\Omega)$.

Problem 148 below says that any continuous function on $\mathbb{R}$ can be approximated uniformly by an entire function [A result of Carleman]. The next 4 problems are required for Problem 148.

144. Given any continuous function $f : \mathbb{R} \to \mathbb{C}$ there is an entire function g such that g has no zeros and $g(x) > |f(x)|\ \forall x \in \mathbb{R}$.

145. Let $f : \mathbb{R} \to \mathbb{C}$ be continuous. Then we can write f as $\sum_{n=-\infty}^{\infty} f_n(x - n)$ where each f_n is continuous and $f_n(x) = 0$ if $|x| \geq 1$.

146. Let $f : \mathbb{R} \to \mathbb{C}$ be continuous and $f(x) = 0$ for $|x| \geq 1$. Let $S = \{z : |\text{Re}(z)| > 3 \text{ and } |\text{Re}(z)| > 2\,|\text{Im}(z)|\}$. Given $\epsilon > 0$ we can find an entire function g such that $|f(x) - g(x)| < \epsilon\ \forall x \in \mathbb{R}$ and $|g(z)| < \epsilon\ \forall z \in S$.

147. Let $f : \mathbb{R} \to \mathbb{C}$ be continuous. Then there is an entire function g such that $|f(x) - g(x)| < 1 \ \forall x \in \mathbb{R}$.

148. Let $f : \mathbb{R} \to \mathbb{C}$ and $\eta : \mathbb{R} \to (0, \infty)$ be continuous. Then there is an entire function g such that $|f(x) - g(x)| < \eta(x) \ \forall x \in \mathbb{R}$.

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149. [Used in problem 146) above]

Let $a < b$ and $f : [a, b] \to \mathbb{C}$ be continuous. Let $f_n(x) = \frac{n}{\sqrt{2\pi}} \int\limits_a^b e^{-n^2(x-t)^2} f(t)dt$.

Then $f_n(x) \to f(x)$ uniformly on $[a + \delta, b - \delta]$ and $f_n(x) \to 0$ uniformly on $\mathbb{R} \backslash [a - \delta, b + \delta]$ for each $\delta > 0$.

150. Show that the family of all analytic maps $f : U \to \{z : \mathrm{Re}(z) > 0\}$ with $|f(0)| \leq 1$ is normal.

151. Let $f \in H(\Omega)$ and f be injective. If $\{z : |z - a| \leq r\} \subset \Omega$ show that

$f^{-1}(z) = \frac{1}{2\pi i} \int\limits_\gamma \frac{\zeta f'(\zeta)}{f(\zeta) - z} d\zeta \ \forall z \in f(B(a, r))$, where $\gamma(t) = a + re^{2it}, 0 \leq t \leq 1$.

152. If $f \in C(\bar{U}) \cap H(U)$ show that $f(z) = i \, \mathrm{Im}(f(0)) + \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} \, \mathrm{Re} \, f(e^{it}) dt$

$\forall z \in U$.

153. If Ω is simply connected show that for any real harmonic function u on Ω, a harmonic conjugate v of u is given by $v(z) = \mathrm{Im}[u(a) + \int\limits_\gamma (\frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y}) dz]$ where a is a fixed point of Ω and γ is any path from a to z in Ω.

154. Let Ω be a region and $f, g \in H(\Omega)$. If $|f| + |g|$ attains its maximum on Ω at some point a of Ω then f and g are both constants.

155. If f and g are entire functions with $f(n) = g(n) \ \forall n \in \mathbb{N}$ and if $\max\{|f(z)|, |g(z)| \leq e^{c|z|}$ for $|z|$ sufficiently large with $0 < c < 1$ show that $f(z) = g(z) \ \forall z \in \mathbb{C}$. Show that the condition $0 < c < 1$ cannot be removed.

156. Show that there is a function f in $C(\bar{U}) \cap H(U)$ whose power series does not converge uniformly on $\bar{U}$.

See also Problem 258 below

157. If $\{f_n\} \subset H(\Omega)$ and $\lim\limits_{n\to\infty} f_n(z) = f(z)$ exists $\forall\ z \in \Omega$ show that there is a dense open subset Ω_0 of Ω such that $f \in H(\Omega_0)$.

158. Let $L : H(\Omega) \to H(\Omega)$ be linear and mulitplicative, not identically 0. Show that there is a point $c \in \Omega$ such that $L(f) = f(c)\ \forall\ f \in H(\Omega)$.

159. Let Ω be a region and $f \in H(\Omega)$ with $f(z) \neq 0\ \forall z \in \Omega$. If f has a holomorphic square root does it follow that it has a holomorphic logarithm? What if it has a holomorphic $k-th$ root for infinitely many positive integers k?

160. Prove that $\lim\limits_{z\to a} \frac{f(z)}{g(z)} = \lim\limits_{z\to a} \frac{f'(z)}{g'(z)}$ if f and g are analytic in some neighbourhood of a, $f(a) = g(a) = 0$ and $g'(a) \neq 0$.

161. If f and g are analytic in some deleted neighbourhood of $a, |f(z)| \to \infty$ and $|g(z)| \to \infty$ as $z \to a$ then $\lim\limits_{z\to a} \frac{f(z)}{g(z)} = \lim\limits_{z\to a} \frac{f'(z)}{g'(z)}$ provided $\lim\limits_{z\to a} \frac{f'(z)}{g'(z)}$ exists.

162. Let f be an entire function such that $|f(z)| = 1$ whenever $|z| = 1$. Show that $f(z) \equiv cz^n$ for some non-negative integer n and some constant c with modulus 1. [Same as Problem 166]

Problem 163

If f is continuous on $\mathbb{C}$ and holomorphic on $\mathbb{C}\backslash T$ does it follow that f is entire? [Recall that $T = \{z : |z| = 1]$.

164. Let Ω be a region (not necessarily bounded) which is not dense in $\mathbb{C}$, $f \in C(\bar{\Omega}) \cap H(\Omega), |f(z)| \leq M\ \forall z \in \partial\Omega$. Suppose f is bounded on Ω. Then $|f(z)| \leq M\ \forall z \in \Omega$.

165. In above problem the hypothesis that Ω is not dense can be deleted provided $\Omega \neq \mathbb{C}$.

166. If f is an entire function such that $|f(z)| = 1$ whenever $|z| = 1$ show that $f(z) = cz^n$ for some $n \geq 0$ and $c \in \mathbb{C}$ with $|c| = 1$.[Same as Problem 162].

167. Let $f \in H(\Omega\backslash\{a, a_1, a_2, ...\})$ where Ω is a region, $a_n \to a$, $a'_n s$ are distinct points of Ω and $a \in \Omega$. If f has a pole at each a_n show that $f(B(a, \epsilon)\backslash\{a, a_1, a_2, ...\})$ is dense in $\mathbb{C}$ for every $\epsilon > 0$.

168. If f is a rational function such that $|f(z)| = 1$ whenever $|z| = 1$ show that $f(z) = cz^n\{\prod\limits_{j=1}^{k} \frac{z-\alpha_j}{1-\bar{\alpha}_j z}\}/\{\prod\limits_{j=1}^{m} \frac{z-b_j}{1-\bar{b}_j z}\}$ for some $n \in \mathbb{Z}$ and $a_1, a_2, ..., a_N, b_1, b_2, .., b_m \in \mathbb{C}\backslash T, c \in \mathbb{C}$ with $|c| = 1$.

169. Let f and g be holomorphic on U with g one-to-one and $f(0) = g(0) = 0$, If $f(U) \subset g(U)$ show that $f(B(0,r)) \subset g(B(0,r))$ for any $r \in (0,1]$.

170. It is a standard fact that injective holomorphic maps from U onto itself are of the type $c\frac{z-a}{1-\bar{a}z}$ with $|a| < 1, |c| = 1$. Find all $m - to - 1$ holomorphic maps of U onto itself for a given positive integer m.

171. Let Ω_1 and Ω_2 be bounded regions. Let $f : \Omega_1 \to \Omega_2$ be a holomorphic map such that there is no sequence $\{z_n\}$ in Ω_1 converging to a point in $\partial\Omega_1$ such that $\{f(z_n)\}$ converges to a point in Ω_2. Then there is a positive integer m such that f is $m - to - 1$ on Ω_1.

172. The condition in Problem 171) above that there is no sequence $\{z_n\}$ in Ω_1 converging to a point in $\partial\Omega_1$ such that $\{f(z_n)\}$ converges to a point in Ω_2 is equivalent to the fact that $f^{-1}(K)$ is compact whenever K is a compact subset of Ω_2.

173. Prove that the analogue of Problem 171) when $\Omega_1 = \Omega_2 = \mathbb{C}$ and $\partial\Omega_1$ is interpreted as (the boundary in $\mathbb{C}_\infty$ i.e.) $\{\infty\}$ holds. Give an example to show that Problem 171) fails for a general unbounded region Ω_1.

174. Let $f \in H(U), \theta_1 \in \mathbb{R}, \theta_2 \in \mathbb{R}$ and $\left|f(re^{i\theta_1})\right| = |f(0)| = \left|f(re^{i\theta_2})\right|$ for all $r \in (0,1)$. Show that f is a constant if $\frac{\theta_1 - \theta_2}{2\pi}$ is irrational.

175. Suppose $\theta_1 \in \mathbb{R}, \theta_2 \in \mathbb{R}$ and $f \in H(U)$, $\left|f(re^{i\theta_1})\right| = |f(0)| = \left|f(re^{i\theta_2})\right|$ for all $r \in (0,1)$ implies that f is a constant. Show that $\frac{\theta_1 - \theta_2}{2\pi}$ is irrational.

176. A second order differential equation: let Ω be a convex region and $g \in H(\Omega)$. Show that any holomorphic function f satisfying the differential equation $f'' + f = g$ in Ω can be expressed as $h(z)\sin(z) + \phi(z)\cos(z)$ for suitable $h, \phi \in H(\Omega)$.

177. Show that $U\backslash\{0\}$ is not conformally equivalent to $\{z : 1 < |z| < 2\}$.

178. Let f be continuous on $\{z : |z| \le R\}$ and holomorphic on $B(0,R)$. Let $M(r) = \sup\{|f(z)| : |z| = r\}$ and $\phi(r) = \sup\{\operatorname{Re} f(z) : |z| = r\}$ for $0 \le r \le R$. Show that $M(r) \le \frac{R-r}{R+r}\operatorname{Re} f(0) + \frac{2r}{R+r}\phi(r)$ and $M(r) \le \frac{R-r}{R+r}|f(0)| + \frac{2r}{R+r}\phi(R)$ for $0 \le r \le R$.

179. If f is an entire function such that $\operatorname{Re} f(z) \leq B|z|^n$ for $|z| \geq R$ then f is a polynomial of degree at most n.

180. Let Ω be a region and A be a subset of Ω with no limit points in Ω. Show that $\Omega \backslash A$ is a region.

181. Show that $\mathbb{C} \backslash (Q \times Q)$ is connected. [See also Problem 229]

182. Prove the formula $\int\limits_{-\infty}^{\infty} e^{itx} e^{-x^2/2} dx = \sqrt{2\pi} e^{-t^2/2} (t \in \mathbb{R})$ in four different ways.

183. Prove that $|e^z - 1 - z| \leq \frac{|z|^2}{2} e^{|z|}$ $\forall z \in \mathbb{C}$ and $|e^z - 1 - z| \leq \frac{|z|^2}{2}$ if $\operatorname{Re}(z) = 0$. Also show that $\left|e^z - 1 - z - z^2/2! - ... - z^n/n!\right| \leq \frac{|z|^{n+1}}{(n+1)!} e^{|z|}$ $\forall z \in \mathbb{C}$.

184. Let f be a non-constant entire function. Show without using Picard's Theorem that $\liminf\limits_{|z| \to \infty} |f(z)| \in \{0, \infty\}$.

185. Let Ω be open and $f \in H(\Omega)$ be one-to-one. Let γ be any closed path in Ω and $\Omega_1 = \{z \in \Omega \backslash \gamma^* : Ind_\gamma(z) \neq 0\}$. Show that $f^{-1}(w) Ind_\gamma(f^{-1}(w)) = \frac{1}{2\pi i} \int\limits_\gamma \frac{z f'(z)}{f(z) - w} dz$ $\forall w \in f(\Omega_1)$.

186. Let $f \in H(U \backslash \{0\})$ and assume that f has an essential singularity at 0. Let $f_n(z) = f(\frac{z}{2^n}), n \geq 1, z \in U \backslash \{0\}$. Show that $\{f_n\}$ is not normal in $H(U \backslash \{0\})$.

187. Let Ω be an open set in $\mathbb{C}$ such that $\mathbb{C}_\infty \backslash \Omega$ is connected. Let γ be closed path in Ω. Show that $Ind_\gamma(a) = 0$ $\forall a \in \mathbb{C} \backslash \Omega$.

188. If f is an entire function which is not a translation show that $f \circ f$ has a fixed point.

Remark: of course, f itself need not have a fixed point: e.g., $z + e^z$.

189. Show that there is a sequence of polynomials $\{p_n\}$ such that $\lim\limits_{n \to \infty} p_n(z) =$

$$\begin{cases} 0 \text{ if } \operatorname{Im}(z) = 0 \\ 1 \text{ if } \operatorname{Im}(z) > 0 \\ -1 \text{ if } \operatorname{Im}(z) > 0 \end{cases}$$

See also Problem 245 below.

190. Show that there is a sequence of polynomials $\{p_n\}$ such that $\lim\limits_{n\to\infty} p_n(z) = 0\ \forall z \in \mathbb{C}$ but the convergence is not uniform on at least one compact set.

191. If A is bounded in $\mathbb{C}$ then $\mathbb{C}_\infty\backslash A$ is connected if and only if $\mathbb{C}\backslash A$ is connected. If A is unbounded and $\mathbb{C}\backslash A$ is connected does it follow that $\mathbb{C}_\infty\backslash A$ is connected? If $\mathbb{C}_\infty\backslash A$ is connected does it follow that $\mathbb{C}\backslash A$ is connected?

192. Let Ω be a bounded region, $a \in \Omega$ and $f : \Omega \to \Omega$ be a holomorphic map such that $f(a) = a$. Show that $|f'(a)| \leq 1$.

193. Let $f \in H(U\backslash\{0\})$ and $|f(z)| \leq \log\frac{1}{|z|}\ \forall z \in U\backslash\{0\}$. Show that f vanishes identically.

194. Let f be an entire function with $|x|\,|f(x+iy)| \leq 1\ \forall x, y \in \mathbb{R}$ then $f(z) = 0\ \forall z \in \mathbb{C}$.

195. Let $f_n : U \to U$ be holomorphic and suppose $f_n(0) \to 1$. Show that $\{f_n\}$ converges to 1 uniformly on compact subsets of U.

196. If $n \in \{3, 4, ...\}$ show that the equation $z^n = 2z - 1$ has a unique solution in U.

197. Show that there are (restrictions to $\mathbb{R}$ of) entire functions which tend to ∞ (as $x \to \infty$) faster than any given function. More precisely, if $\phi : (0,\infty) \to (0,\infty)$ is any increasing function then there is an entire function f such that $f(x) \geq \phi(x)\ \forall x \in (0,\infty)$.

198. Find a necessary and sufficient condition that $A \equiv \{z : \left|az^2 + bz + c\right| < r\}$ is connected.

199. If $z, c_1, c_2, c_3 \in \mathbb{C}$ and $\frac{1}{z-c_1} + \frac{1}{z-c_2} + \frac{1}{z-c_3} = 0$ show that z belongs to the closed triangular region with vertices c_1, c_2, c_3.

200. Prove the following result of Gauss and Lucas: if p is a polynomial then every zero of p' is in the convex hull of the zeros of p.

201. Let $f \in C(\bar{U}) \cap H(U)$. Show that $\int\limits_{-1}^{1} |f(x)|^2 \, dx \leq \sqrt{\int\limits_{0}^{\pi} |f(e^{it})|^2 \, dt} \sqrt{\int\limits_{0}^{\pi} |f(e^{-it})|^2 \, dt}$.

202. Prove Brouer's Fixed Point Theorem in two dimensions: every continuous map $\phi : \bar{U} \to \bar{U}$ has a fixed point.

203. If $\phi : T \to \mathbb{C} \backslash \{0\}$ is continuous and if $\phi(-z) = -\phi(z) \ \forall z \in T$ show that there is no continuous function g on T such that $g^2 = \phi$.

204. Prove that if K is a non-empty compact convex subset of $\mathbb{C}$ then every continuous map $\phi : K \to K$ has a fixed point.

205. If $f \in H(B(0,\delta)), f(0) = 0$ and $f(z) \neq 0 \ \forall z \in B(0,\delta) \backslash \{0\}$ show that $|f(z)|$ is not harmonic. (Example: $|z|^n$)

206. Prove Rado's Theorem

Let Ω be a region, $f \in C(\Omega)$ and $f \in H(\Omega_0)$ where $\Omega_0 = \Omega \backslash f^{-1}\{0\}$. Then $f \in H(\Omega)$

207. Let $f \in H(\mathbb{C} \backslash \{0\})$ and suppose f does not have an essential singularity at 0. If $f(e^{it}) \in \mathbb{R} \ \forall t \in \mathbb{R}$ show that $f(z) = \frac{p(z)}{z^k}$ for some non-negative integer k and some polynomial p whose degree does not exceed $2k$.

208 Find a necessary and sufficient condition that $az^2 + bz + c$ (with $a \neq 0$) is one-to-one in U.

209 Let $c_1, c_2, ..., c_n$ be distinct complex numbers. Show that $\sum\limits_{k=1}^{n} \prod\limits_{j \neq k} \frac{c_j - c}{c_j - c_k} = 1$ for all $c \in \mathbb{C}$.

210.

Let μ be a finite positive measure on the Borel subsets of $(0, \infty)$. If $g \in L^\infty(\mu)$ and $\int\limits_{0}^{\infty} e^{-x} p(x) g(x) d\mu(x) = 0$ for every polynomial p show that $g = 0$ a.e. $[\mu]$. Conclude that $\{e^{-x}p(x) : p \text{ is a polynomial}\}$ is dense in $L^1(\mu)$.

211.

Let $\Omega = \mathbb{C} \backslash \{0, 1\}$ and $f : \Omega \to \Omega$ be holomorphic. Show that if f is not a constant then it must be one of four specific Mobius transformations. [Proposed and solved by Walter Rudin in Amer. Math. Monthly]

212.

Let g be an entire function and $f : \mathbb{R} \to \mathbb{R}$ be a C^∞ function such that for each real number x there is a positive integer $n(x)$ with $f(x) = [g(x)]^{n(x)}$. Show that $f \equiv g^n$ for some positive integer n independent of x.

213.

a) What is $T + T$?. Is the representation $z = z_1 + z_2, z_1 \in T, z_2 \in T$ unique for $z \in T + T$?

b) What is $\bigcup_{n=2}^{\infty} \{T + T + ... + T(\text{ n terms})\}$? Do we have uniqueness here?

c) If X is an inner product space of dimension greater than 1 prove that $\{x : \|x\| \leq 2\} = \{y + z : \|y\| = \|z\| = 1\}$.

[See also Problem 269]

214.

Let $|z_j| \leq 1$ for $1 \leq j \leq n$. Show that we can write $\prod_{j=1}^{n}(1 + z_j)$ as $(1 + c)^n$ for some complex number c with $|c| \leq 1$.

215.

Let $f : U \to U$ be a non-constant analytic function. Show that $|f(c)| = |c|$ for some $c \in U$.

216.

Let g be an entire function such that $g(z+\alpha) = ag(z)$ and $g(z+\beta) = bg(z)$ for all z, where $a, b, \alpha, \beta \in \mathbb{C}$ and α, β are independent over $\mathbb{R}$. Give an elementary proof (not using elliptic functions) to show that g has the form ce^{dz} for some constants c and d.

217

Let $a_n = \frac{(2n-2)!}{2^{2n-1}n!(n-1)!}, n = 1, 2, ...$ and $a_0 = -1$. Show that $e^{-\frac{1}{2}Log(1-z)} = -\sum_{n=0}^{\infty} a_n z^n$ for $|z| < 1$. Show that the power series $\sum_{n=0}^{\infty} a_n z^n$ actually converges absolutely for $|z| \leq 1$ and that $\sum_{n=0}^{\infty} a_n = 0$. Use this to compute $\sum_{k=0}^{n} a_k a_{n-k}, n = 1, 2, ...$

Problem 218

Let $\gamma : [0, 1] \to \mathbb{C}$ be continuous, $\gamma(1) = \gamma(0)$, $z \in \mathbb{C}\backslash\gamma([0, 1])$ and $\phi_z : [0, 1] \to \mathbb{C}$ be a continuous function such that $e^{\phi_z(t)} = \gamma(t) - z, 0 \leq t \leq 1$. [

Such a function always exists and $\frac{\phi_z(1)-\phi_z(0)}{2\pi i}$ is called the index of z w.r.t. γ]. Show that $\phi_z(1) - \phi_z(0) \to 0$ as $|z| \to \infty$.

Problem 219

Show we cannot have $a > 0$ and $f \in H(\mathbb{C}\backslash\{0\})$ such that $|f(z)| \geq a\,|z|^{-1/2}$ $\forall z \in \mathbb{C}\backslash\{0\}$.

Problem 220

Give a simple example of a function f in $C(\bar{U}) \cap H(U)$ such that f does not extend to a function in $H((1+\varepsilon)U)$ for any $\varepsilon > 0$.

Problem 221

Let Ω be a region, $c \in \Omega$ and let R_c be the radius of the largest open ball around c which is contained in Ω. Show that $|R_c - R_d| \leq |c - d|$ for all $c, d \in \Omega$.

Problem 222

Let $\Omega = \{z \in \mathbb{C} : 1 < |z| < 3\}$ and γ be the positively oriented circle with center 0 and radius 2. Let $f \in H(\Omega)$ and $\int\limits_{\gamma} f(z)dz = 0$. Show that there exists $g \in H(\Omega)$ with $g'(z) = f(z)$ $\forall z \in \Omega$. [Hence $\int\limits_{\gamma_1} f(z)dz = 0$ for every closed path γ_1 in Ω!]. Give an example to show that f may satisfy above property without being a second derivative in Ω. If, for each positive integer n, there is a function f_n in $H(\Omega)$ such that $f = f_n^{(n)}$ (the $n-th$ derivative of f_n) show that f extends to a holomorphic funtion on $\{|z| < 3\}$.

Problem 223

Let f, g, h be entire functions such that $|f| = |g| + |h|$ everywhere. What relations exist between these functions?

Problem 224

Show that there is no region Ω in which there exist non-constant holomorphic functions f and g with $|f(z)| = \operatorname{Re} g(z)$ $\forall z \in \Omega$.

Problem 225

Let A be an $n \times n$ complex matrix with distinct non-zero eigen values. Show that there is a matrix B such that $\sum\limits_{k=0}^{\infty} \frac{B^k}{k!}$ converges to A.

Problem 226

Let f be an entire function such that $f(0) = 0, f$ not identically 0 and f maps the real line into itself. Suppose the image of the imaginary axis under f is contained in a straight line L. Show that L must be one of the coordinate axes.

Problem 227

Find all meromorphic functions f on $\mathbb{C}$ such that $|f(z)| = 1$ whenever $|z| = 1$.

Problem 228

Problem 229

Let Ω be an open connected set in $\mathbb{R}^n$. If $A \subseteq \Omega$ is at most countable then $\Omega \backslash A$ is connected.

Problem 230

Let $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$ have radius of convergence 1 and suppose $a_n \geq 0$ for all n. Show that f cannot extend to a holomorphic function on $U \cup B(1, \delta)$ for any $\delta > 0$.

Problem 231

Let $\Omega = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ and f be a bounded holomorphic function on Ω. Show that f is uniformly continuous on $\{z \in \mathbb{C} : \operatorname{Re} z > \delta\}$ for any $\delta > 0$.

Problem 232

If f and g are non-constant entire functions such that $f \circ g$ is a ploynomial prove that f and g are both polynomials.

Problem 233

Find all entire functions f such that $f(2z) = (1 - 2z)f(z)$ for all z.

Problem 234

Let $p_n(z) = 1 + \frac{z}{1!} + \frac{z^2}{2!} + ... + \frac{z^n}{n!}$ where n is a positive integer ≥ 2. Show that p_n has n distinct zeros $c_1, c_2, ..., c_n$ which are non-zero and satisfy the relation $\sum_{j=1}^{n} \frac{1}{c_j^k} = 0$ for $2 \leq k \leq n$.

Problem 235

Let $f : U \to \bar{U}$ be holomorphic. Show that $|f'(0)| \leq 1$ and equality holds if and only if there is a real number t such that $f(z) = e^{it}z$ for all z.

[Thus, the condition $f(0) = 0$ in Schwartz Lemma is redundant].

Problem 236

Let Ω be a bounded region in $\mathbb{C}$ and $f : \Omega \to \Omega$ be holomorphic. Suppose $c \in \Omega$ and $f(c) = c$. Show that $|f'(c)| \leq 1$.

Problem 237

In Problem 236 above suppose $f'(c) = 1$. Show that f must be the identity map on Ω.

Problem 238

Let Ω be a bounded region, $f : \Omega \to \Omega$ a conformal equivalence (onto Ω). If $c \in \Omega, f(c) = c$ and $f'(c) \in (0, \infty)$ then f is the identity map.

Problem 239

Prove Poincare's Uniqueness Theorem (assuming Riemann Mapping Theorem):

if $\Omega \neq \mathbb{C}$ is simply connected then there is a unique conformal map f of Ω onto U such that $f(\zeta_0) = z_0$ and $\frac{f'(\zeta_0)}{|f'(\zeta_0)|} = e^{it}$ for any given $\zeta_0 \in \Omega, z_0 \in U$ and $t \in \mathbb{R}$.

Problem 240

Let Ω be a bounded region and $f : \Omega \to \Omega$ be holomorphic. Let $f_1 = f, f_{n+1} = f \circ f_n$ for $n \geq 1$. Let $c \in \Omega$ and $f(c) = c$. Prove that either a subsequence of f_n converges to c uniformly on compact sets or else f is a homeomorphism (hence a conformal equvalence) of Ω onto itself.

[See also Problem 241 below].

Problem 241

In previous problem either f is a conformal equivalence of Ω onto itself or else $\{f_n\}$ converges uniformly on compact sets to c.

Problem 242

Let Ω be a bounded region and $f : \Omega \to \Omega$ be holomorphic. Let $c_1, c_2 \in \Omega, c_1 \neq c_2, f(c_1) = c_1$ and $f(c_2) = c_2$.

Show that f is a conformal equivalence of Ω.

Problem 243

Show that $\{z : r_1 < |z - c_1| < R_1\}$ is conformally equivalent to $\{z : r_2 < |z - c_2| < R_2\}$ (where $c_1, c_2 \in \mathbb{C}, 0 < r_1 < R_1$ and $0 < r_2 < R_2$) if and only if $\frac{R_1}{r_1} = \frac{R_2}{r_2}$.

Problem 244

Prove that $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$ is an enire function of order 0 if $|a_n| \leq \frac{1}{(n!)^n}$ for n sufficiently large.

Problem 245

Show that any measurable function $f : \mathbb{C} \to \mathbb{C}$ is the almost everywhere limit (w.r.t. two dimensional Lebesgue measure) of a sequence of polynomials.

Problem 246

If Ω is a region and $f, f_1, f_2, ...$ are continuous functions such that each f_n is holomorphic and $f_n \to f$ pointwise on Ω show that there is a dense open set Ω_1 in Ω such that f is holomorphic in Ω_1.

Problem 247

Define a sequence of complex numbers $\{\alpha_n\}$ by $\alpha_1 = i$ and $\alpha_{n+1} = i^{a_n} \equiv e^{\alpha_n Log(i)}$. Show that $\{\alpha_n\}$ converges.

Problem 248

Let Ω_1 and Ω_2 be simply connected regions such that $\Omega_1 \cap \Omega_2$ is non-empty and connected. Show that $\Omega_1 \cap \Omega_2$ and $\Omega_1 \cup \Omega_2$ are both simply connected. Give an example to show that $\Omega_1 \cup \Omega_2$ can be connected but not simply connected if the assumption that $\Omega_1 \cap \Omega_2$ is connected is dropped.

Problem 249

Show that $\{n(\int\limits_0^n \frac{\sin x}{x} dx - \frac{\pi}{2})\}$ is bounded.

Problem 250 [Morera's Theorem for rectangles]

Let $f \in C(U)$ and $\int\limits_\gamma f = 0$ for every rectangular path γ in U. Show that $f \in H(U)$.

Problem 251

Morera's Theorem for circles

Let f be a continuous function on $\mathbb{C}$ such that $\int\limits_\gamma f = 0$ for every circle γ. Then f is entire.

Problem 252 [Anti-calculus proposition of Polya, Szego and Erdos]

Let f be holomorphic in an open set containing the closed unit disc and c be a point on the unit circle such that $|f(c)| = \sup\{|f(z)| : |z| \leq 1\}$. Then $f'(c) \neq 0$ unless $f \equiv 0$.

Problem 253

Let γ_1, γ_2 be two closed paths in $\mathbb{C}\backslash\{0\}$ such that $Ind_{\gamma_1}(0) = Ind_{\gamma_2}(0)$. Show that γ_1 and γ_2 are homotopic in $\mathbb{C}\backslash\{0\}$.

Problem 254

Suppose f is holomorphic in a neighbourhood of c and $\sum\limits_{k=1}^{\infty} f^{(k)}(c)$ converges. Show that f extends to an entire function such that $\sum\limits_{k=1}^{\infty} f^{(k)}$ converges uniformly on compact sets.

Problem 255

Show that there exist complex numbers $c_1, c_2, ...$ such that $\sum_{n=0}^{\infty} c_n^k$ converges and $\sum_{n=0}^{\infty} |c_n|^k = \infty$ for every positive integer k. Show that if $\operatorname{Re} c_n \geq 0 \ \forall n$ then convergence of $\sum_{n=0}^{\infty} c_n^k$ for $k = 1$ and $k = 2$ implies that of $\sum_{n=0}^{\infty} |c_n|^2$.

Problem 256

Let ϕ be a homeomorphism from $\{z : |z - a| = r\}$ onto $\{z : |z - c| = \rho\}$. Show that ϕ extends to a homeomorphism of $\mathbb{C}$ onto itself.

Problem 257

Show that $\Omega = U \backslash [0, 1)$ is simply connected. Show also that there is no conformal equivalence from Ω onto U which extends to a homeomorphism from $\bar{\Omega}$ onto $\bar{U}$.

Remark: Ω is an example of a simply connected region which is not a Jordan region. In the case of Jordan regions conformal equivalences with U can always be extended to the closures homeomorphically.

Problem 258

Let $f \in C(\bar{U}) \cap H(U)$ and $\varepsilon > 0$. Show that there is a polynomial p such that $|p(z) - f(z)| < \varepsilon \ \forall z \in \bar{U}$.

Problem 259

Let p and q be non-zero polynomials with rational coefficients such that there is no polynomial with rational coefficients which not a constant and divides both p and q. Show that there exist polynomials p_1 and q_1 with rational coefficients such that $pp_1 + qq_1 \equiv 1$.

Problem 260

Let $a, b, c, d \in T, a \neq b, c \neq d$. There exists a conformal equivalence f of U which extends to a homeomorphism of $\bar{U}$ mapping T onto T such that $f(a) = c$ and $f(b) = d$.

Problem 261

Prove the following version of Maximum Modulus Principle (due to Lindelöf): let Ω be open in $\mathbb{C}, z_0 \in \Omega, r > 0$. Suppose $(\mathbb{C} \backslash \bar{\Omega}) \cap \{z : |z - z_0| = r\}$ contains a closed arc of the circle $\{z : |z - z_0| = r\}$ of length $\frac{2\pi r}{k}$ where $k \in \mathbb{N}$. Suppose $f \in H(\Omega)$ and $|f(z)| \leq M \ \forall z \in \Omega$. If $\limsup\limits_{z \to c} |f(z)| \leq \varepsilon$ $\forall c \in B(z_0, r) \cap \partial\Omega$ show that $|f(z_0)| \leq (\varepsilon M^{k-1})^{1/k}$.

Problem 262

Let u be subharmonic on a bounded region Ω. Suppose $\limsup\limits_{z \to c} u(z) \leq M$ for all $c \in \partial\Omega \backslash F$ where F is a finite subset of $\partial\Omega$ and $\limsup\limits_{z \to c} u(z) < \infty$ for $c \in F$. Then $u \leq M$ in Ω.

Problem 263

Let Ω be a bounded region, $F \subseteq \partial\Omega$ a finite set, $f \in H(\Omega), \limsup\limits_{z \to c} |f(z)| \leq M$ for all $c \in \partial\Omega \backslash F$ and $\limsup\limits_{z \to c} |f(z)| < \infty$ for $c \in F$. Then $|f| \leq M$ in Ω.

[This generalizes Problem 59]

Problem 264

Does there exist a continuous closed curve γ in $\mathbb{C}$ such that $Ind_\gamma(z) = 0$ for every $z \in \mathbb{C} \backslash \gamma^*$?

Problem 265

Prove the following version of Schwartz Lemma:

$f \in H(U), |f(z)| \leq 1 \ \forall z \in U, f(0) = 0, f(z) = \sum\limits_{k=0}^{\infty} a_k z^k \ (z \in U)$ implies $\sum\limits_{k=0}^{\infty} |a_k|^2 \leq 1$. If $|a_k| = 1$ then $f(z) = cz^k$ for some constant $c \in T$.

Remark: note that $f'(0) = a_1$ so we have $|f'(0)| \leq 1$ and equality holds if and only if $f(z) = cz$ for some constant $c \in T$.

Problem 266

Let $f \in H(U)$ and let the radius of convergence of the power series $\sum\limits_{k=0}^{\infty} a_k z^k$ of f be 1. If $a_n \to 0$ as $n \to \infty$ and f can be extended to holomorphic function on $U \cup B(e^{ia}, r)$ for some $r > 0$ and $a \in \mathbb{R}$ show that f has either a removable singularity or an esential singularity at e^{ia}.

Problem 267

Show that the series $\sum_{n=0}^{\infty} \frac{1}{2^n} \frac{z^n}{1-z^n}$ converges for $|z| < 1$ as well as for $|z| > 1$ but it does not extend to an entire function.

Remark: it can be shown that T is a natural boundary for $\sum_{n=0}^{\infty} \frac{1}{2^n} \frac{z^n}{1-z^n} (|z| < 1)$.

Problem 268

Show that T is a natural boundary for $f(z) = \sum_{n=0}^{\infty} z^{n!}$.

Problem 269

Prove or disprove that any complex number can be expressed as a sum of a finite number of *distinct* complex numbers on the unit circle T.

[See also Problem 213]

Problem 270

If f and $g \in H(U), |f| \leq |g|$ and g extends to a holomorphic function on an open set strictly containing U can we say the same thing about f?.